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Generalized Dyck equations and multilabel trees

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Abstract

New topological operations are introduced in order to recover the generalized Dyck equations presented by D. Arquès et al. in another way for the generating functions of maps and colored maps, by decomposing maps topologically and bijectively. By repeatedly applying the operations which made it possible to reveal the generalized Dyck equations for the successive transformed maps, one-to-one correspondences are obtained between maps (colored or not) of indeterminate genus and trees (colored or not) whose vertices can be labelled with several labels, following rules that we will define. These bijections provide us with a coding of these maps.

Résumé

De nouvelles opérations topologiques sont introduites afin de nous permettre de retrouver les équations de Dyck généralisées aux cartes (coloriées ou non) de genre quelconque données par D. Arquès et al., par des méthodes topologiques et bijectives de décomposition des cartes. En appliquant plusieurs fois les opérations qui nous ont permis de retrouver les équations de Dyck généralisées aux cartes successives obtenues, on obtient des bijections entre cartes (coloriées ou non) de genre quelconque et des arborescences (coloriées ou non) où les sommets peuvent être étiquetés par plusieurs étiquettes suivant des règles que nous définirons. Ces bijections nous fournissent un codage de ces cartes.

Keywords: Map; Colored map; Dyck equation

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1. Introduction

The enumerative study of maps started in 1963 with W. Tutte, who enumerated rooted planar maps first with n edges [17], and then in 1968 with v vertices and f faces [18]. Walsh and Lehman [19] obtained an algorithm for counting rooted maps of orientable genus g with v vertices and f faces. Maps can also be described as combinatorial objects [10,13]. In 1975, Cori [7] studied planar maps in this perspective and extended these results with Machi [8] to orientable maps. In 1987, Arquès [1] determined functional relations satisfied by generating functions of rooted maps on the torus and obtained closed formulas to enumerate these maps by vertices and faces. Several studies followed on maps of greater genus, orientable or not, as for example the papers of Bender and Canfield [6] and also Arquès and Giorgetti [4].

The study of rooted maps independent of their genus began with Walsh and Lehman [19]. They gave a recursive relation on the number of rooted maps with respect to the number of edges, which does not lead to an explicit enumeration formula of these maps. In 1990, Jackson and Visentin [12] used an algebraic approach and obtained a closed formula for the generating functions of orientable rooted maps with respect to the number of edges and vertices.

More recently, Arquès and Béraud [2,3] and Arqués and Micheli [5] determined a functional equation satisfied by the generating functions of rooted maps (respectively, colored maps) with respect to the number of edges and vertices, that generalizes the Dyck equation on trees, and express the solution in a continued (respectively, multi-continued) fraction form. The continued fraction reveals an interesting bijection, since it also enumerates connected fixed-point free involutions. Ossona de Mendez and Rosenstiehl [16] describe this bijection. From the combinatorial structure that they gave for rooted maps, they deduced a code for each map with a connected fixed-point free involution.

Topological operations applied to a map, such as the removal or the addition of an edge or the fusion of two vertices, sometimes modify the genus of the map. These operations therefore cannot be carried out in a systematic way when one works with fixed genus. However, these elementary operations make it possible to find new functional equations on maps counted independent of genus and to establish bijections between families of maps.

In Section 2, we recall general definitions about maps. New topological operations are introduced in Section 3 in order to establish, in Section 4, a bijection between maps of indeterminate genus and maps of indeterminate genus with a root bridge edge, in which a subset of their vertices has been selected. This bijection provides us with a new proof of the generalized Dyck equation on orientable rooted maps given by Arquès and Béraud [2,3]. They obtain this equation by an analytic resolution of a differential equation satisfied by the generating function that counts rooted maps. Here we present a new proof of this equation, without any transformation on the generating function, but only by transcription of the given bijection. Flajolet [11], moreover, showed that many continued fractions having integer coefficients can be explained in a purely combinatorial way, and here is another instance of this assertion.

In Section 5, we then give a bijection between orientable rooted maps and a family of trees whose vertices can be labelled by several labels according to certain rules, which is deduced from the one presented in Section 4 by successive applications of this bijection. A generalization of the bijection between planar maps and well-labelled trees [9] to maps of genus g and well-labelled g-trees [14] allowed M. Marcus and B. Vauquelin to obtain a code for maps of genus g by words product, i.e. each map is encoded by a shuffle of a Dyck word with constraints and a sequence of integers. The bijection enables us to determine a new language coding the maps of indeterminate genus.

We finally extend these results to n-colored rooted maps of indeterminate genus in Section 6.2. The bijection between n-colored rooted maps and n-colored rooted maps with a root bridge in which a subset of these vertices has been selected provides us with a new proof of the generalized Dyck equation on orientable n-colored rooted maps obtained formerly in an analytical way [5].

2. Definitions

We recall some definitions used in the sequel (for further details, see for example [7,8]).

A *topological map* C in an orientable surface Σ in $\mathbb{R}^3$ is a partition of Σ into three finite sets of cells:

(1) The set of vertices of C, which is a finite set of points;
(2) The set of edges of C, which is a finite set of open Jordan arcs, pairwise disjoint, whose extremities are vertices (a loop is a closed Jordan arc one of whose points is the incident vertex); and
(3) The set of faces of C. Each face is simply connected and its border is the union of the vertices and edges incident to it.

The *genus* of the map C is the genus of Σ. A cell is *incident* to another cell if one is contained in the boundary of the other. A *bridge* is an edge incident on both sides to the same face. We call a *half-edge* an oriented edge of the map.

Let B be the set of half-edges of the map. With each half-edge, one can associate its initial vertex, its terminal vertex and its underlying edge. We denote by α (resp. σ) the permutation on B that takes each half-edge b into its opposite half-edge (resp. the first half-edge encountered when turning round the initial vertex of b in the positive direction according to the orientation imposed on the surface). The cycles of α (resp. σ) represent the edges (resp. the vertices) of the map. The cycles of $\bar{\sigma} = \sigma \circ \alpha$ are the oriented borders of the faces of the map. (B, σ, α) is the *combinatorial definition* of the topological orientable map-associated C. We note here that the group generated by σ and α is transitive on B.

A map $C = (B, \sigma, \alpha)$ is *rooted* if a half-edge $\tilde{b}$ is distinguished. The half-edge $\tilde{b}$ is called the *root half-edge* of C, and its initial vertex is the *root vertex*. C is then defined as the triplet $(\sigma, \alpha, \tilde{b})$. The face $\bar{\sigma}^*(\tilde{b})$ is called the *exterior face* of C. By convention, the vertex-map (one vertex, no edges) is said to be rooted.

Two orientable maps of the same genus are *isomorphic* if there is a homeomorphism of the surfaces, preserving its orientation, mapping vertices, edges and faces of one map onto vertices, edges and faces, respectively, of the other map. An isomorphism class of orientable rooted maps of genus g will simply be called an orientable rooted map.

We denote the vertex-map by $\{p\}$, $\mathscr{M}$ as the set of orientable rooted maps, $\mathscr{I}$ as the subset of $\mathscr{M}$ of maps with a bridge root edge, and for any map $I \in \mathscr{I}$, $\mathrm{Right}(I)$ (resp.

Left(I)) the maximal submap of I containing the root vertex (resp. the terminal vertex of $\tilde{b}$) such that the root half-edge $\tilde{b}$ (resp. $\alpha(\tilde{b})$) of I does not belong to Right(I) (resp. Left(I)) (see Fig. 3).

3. Preliminaries

In Section 3.1, we describe two algorithms that number the half-edges and vertices of a map. Numbering induces an order relation on half-edges and vertices that allows us to define, in Section 3.2, new topological operations on maps. These operations will be useful for proving Theorem 11. These two operations are mutual inverses, and they are interesting because one of them (which we call derivation) gathers a subset of vertices of a map into one vertex and the other one (which we call integration) retrieves the original subset of vertices.

3.1. Order relations in a rooted map

Order relations on half-edges and vertices of a map are introduced in this section. We present an algorithm that traverses a map along its half-edges: they are numbered beginning with the root half-edge and in their order of appearance in the oriented circuit given by the algorithm (see map C in Fig. 1. Each number appears near the initial vertex of the half-edge). Half-edges are then naturally ordered by their number.

The root half-edge $\tilde{b}$ acquires number 0; then, the other half-edges of its face, $\bar{\sigma}^*(\tilde{b})$, are numbered. Afterwards while there still are numberless half-edges:

- Among numbered half-edges, the *smallest* half-edge b (the half-edge with the smallest label) with a numberless opposite half-edge is chosen.
- The half-edges of the face $\bar{\sigma}^*(\alpha(b))$ are numbered, beginning with $\alpha(b)$ and proceeding in the order determined by $\sigma \circ \alpha$.

Definition 1 (*Order relation on vertices*). Let C be a rooted map and s_1, s_2 be two vertices of C. The vertex s_1 is *smaller* than s_2 if the smallest half-edge of s_1 is smaller than the smallest half-edge of s_2.

Vertices are numbered by this order relation. Number 1 is assigned to the root vertex and other vertices are numbered in an ascending order such that if vertex v_1 is encountered in the traversal of the map earlier than vertex v_2, its number must be smaller than the number of v_2 (see numbers in bold on map C of Fig. 1).

A map is *ordered* when its half-edges and vertices are numbered by the algorithms given above.

Definition 2 (*Path and subpath of a map*). The *path* of an ordered map C corresponds to the increasing ordered sequence of the half-edges of C, starting from its root half-edge. A *subpath* of C is defined as an increasing subsequence of ordered and successive half-edges of C.

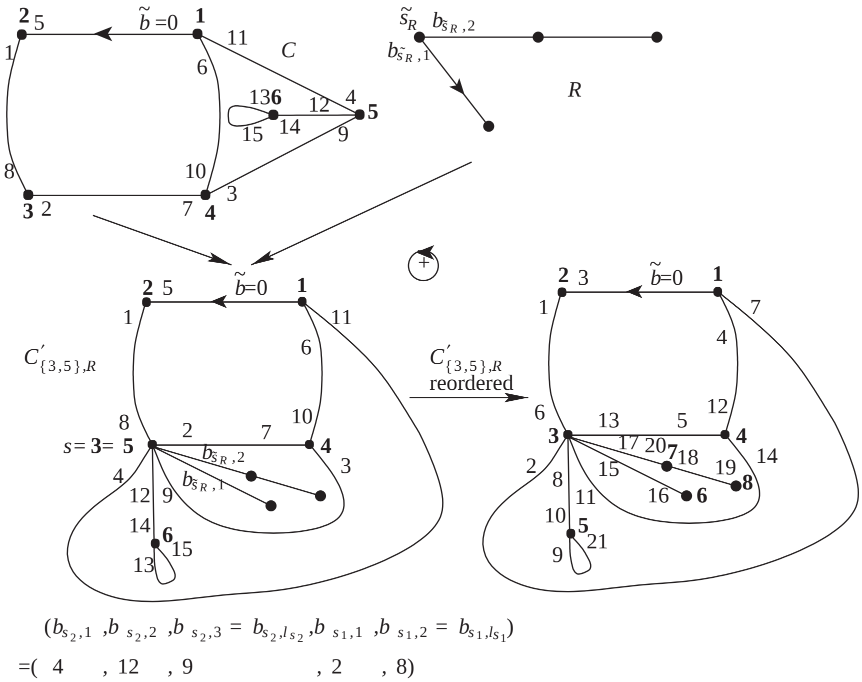

Fig. 1. Derived map with respect to vertices **3** and **5** of a pair of maps.

Lemma 3 (*On the smallest half-edges of a face and of a vertex of an ordered map*). *The smallest half-edge b_s of a vertex s different from the root vertex, of an ordered map $C = (\sigma, \alpha, \tilde{b})$, is not the smallest half-edge of its face $\bar{\sigma}^*(b_s)$.*

The smallest half-edge b_f of a face f different from the exterior face, of an ordered map $C = (\sigma, \alpha, \tilde{b})$, is not the smallest half-edge of its initial vertex.

Proof. If b_s belongs to the exterior face of C, since s is different from the root vertex, we have $\tilde{b} < b_s$ and b_s cannot be the smallest half-edge of its face.

If b_s does not belong to the exterior face of C, the half-edges of face $\bar{\sigma}^*(\alpha(b_s))$ have been numbered before b_s (see the algorithm above). Thus, $\alpha(b_s)$ is smaller than b_s. Then $\bar{\sigma}(\alpha(b_s)) = \sigma(b_s)$, which belongs to vertex s, is smaller than b_s.

Since b_f is the smallest half-edge of face f, the half-edge $\alpha(b_f)$ is smaller than b_f and $\bar{\sigma}(\alpha(b_f)) = \sigma(b_f)$, which belongs to the initial vertex of b_f, is smaller than b_f. $\square$

3.2. Topological and bijective operations on maps

In Section 3.2.1, we define the *derivation* operation that gathers a subset of vertices of a map and the root vertex of a second map into one vertex. These vertices can be recovered by applying the inverse operation, called the *integration* operation and defined in Section 3.2.2, which uses the order properties of a map to retrieve all the gathered vertices. These operations are the main tools used in the proof of Theorem 11.

We denote by $\mathscr{M}_2$ the subset of maps of $\mathscr{M}$ that have at least two distinct vertices.

3.2.1. Derivation of maps

In this section, we define a derived map of a pair of maps (C, R) of $\mathscr{M}_2 \times \mathscr{M}$ with respect to certain vertices of C. To derive a pair of maps with respect to vertices $s_1, \dots, s_m$ of C means to collect these vertices into one vertex while respecting the order of Definition 1 and afterwards to glue this vertex to the root vertex of R, as described in the definition below.

Definition 4 (*Derived map*). Let $C = (\sigma, \alpha, \tilde{b})$ be a map of $\mathscr{M}_2$ with root vertex $\tilde{s}_C$ and $R=(\sigma_R, \alpha)$ be a map of $\mathscr{M}$ with a root vertex $\tilde{s}_R$ and if $R \neq \{p\}$, let $(b_{\tilde{s}_R,1}, b_{\tilde{s}_R,2}, \dots, b_{\tilde{s}_R,l_{\tilde{s}_R}})$ be the half-edges of $\tilde{s}_R$ ordered according to σ_R and $b_{\tilde{s}_R,1}$ be the root half-edge of R. Let $\mathscr{S} = \{s_1, \dots, s_m\}$ be a set of distinct vertices of C such that $\tilde{s}_C < s_1 < s_2 < \cdots < s_m$. For all i in $[1, m]$, let $(b_{s_i,1}, \dots, b_{s_i,l_{s_i}}) = \sigma^*(b_{s_i,1})$ be the half-edges whose initial vertex is s_i, in which $b_{s_i,1}$ is the smallest half-edge of s_i.

The *derived map* $C'_{\mathscr{S},R} = (\sigma', \alpha, \tilde{b})$ of $(C, \mathscr{S}, R)$ is then the map obtained from C and R after gathering the vertices of $\mathscr{S} \cup \{\tilde{s}_R\}$ into a unique vertex s as follows (see Fig. 1):

$$
\begin{aligned}
s &= \left(\underbrace{b_{s_1,1}, \dots, b_{s_1,l_{s_1}}}_{s_1}, \underbrace{b_{s_2,1}, \dots, b_{s_2,l_{s_2}}}_{s_2}, \dots, \underbrace{b_{s_m,1}, \dots, b_{s_m,l_{s_m}}}_{s_m}, \underbrace{b_{\tilde{s}_R,1}, \dots, b_{\tilde{s}_R,l_{\tilde{s}_R}}}_{\tilde{s}_R} \right) \\
&= \sigma'^*(b_{s_1,1}).
\end{aligned}
$$

In terms of permutations, it means: $\sigma' = \tau_{1R}\tau_{1m} \dots \tau_{12}\sigma = \gamma\sigma$ with $\tau_{1i} = (b_{s_1,1}b_{s_i,1})$, $\tau_{1R} = (b_{s_1,1}b_{\tilde{s}_R,1})$ and $\gamma = (b_{s_1,1} \dots b_{s_m,1}b_{\tilde{s}_R,1})$.

Lemma 5 (*Orders of $C'_{\mathscr{S},R}$, of C and of R*). (1) *In $C'_{\mathscr{S},R}$, if $R \neq \{p\}$, $b_{\tilde{s}_R,1}$ is the smallest half-edge among the half-edges of R (see Fig. 1 in which $b_{\tilde{s}_R,1} = 15$).*

(2) *The half-edges smaller than or equal to $\alpha(b_{s_1,l_{s_1}})$ have the same numbering in the ordered maps $C'_{\mathscr{S},R}$ and C.*

Proof. (1) By construction, R is recovered if in $C'_{\mathscr{S},R}$ the subset of half-edges belonging also to R, i.e. $\{b_{\tilde{s}_R,1}, \dots, b_{\tilde{s}_R,l_{\tilde{s}_R}}\}$, is unglued from vertex s. Thus in the traversal of $C'_{\mathscr{S},R}$, starting from its root half-edge, $\tilde{b}$, to reach any half-edge of R, one has to cross s. It implies that there exists i, $1 \leqslant i \leqslant l_{\tilde{s}_R}$, such that $b_{\tilde{s}_R,i}$ is the smallest half-edge of the half-edges of R in $C'_{\mathscr{S},R}$. If $l_{\tilde{s}_R} > 1$, we prove that $b_{\tilde{s}_R,1}$ is the smallest half-edge of the half-edges of R in $C'_{\mathscr{S},R}$.

Now $b_{\tilde{s}_R,i}$ cannot be the smallest half-edge of its face $\overline{\sigma'}^*(b_{\tilde{s}_R,i})$; otherwise $\alpha(b_{\tilde{s}_R,i})$, which belongs to R and has been previously numbered as part of the face $\overline{\sigma'}^*(b_{\tilde{s}_R,i})$, is smaller than $b_{\tilde{s}_R,i}$.

If $i>1$, $b_{\tilde{s}_R,i}=\sigma'(b_{\tilde{s}_R,i-1})=\overline{\sigma'}(\alpha(b_{\tilde{s}_R,i-1}))$, so that $\alpha(b_{\tilde{s}_R,i-1})$, which belongs to R, is smaller than $b_{\tilde{s}_R,i}$ (since $b_{\tilde{s}_R,i}$ is not the smallest half-edge of its face), which contradicts the definition of $b_{\tilde{s}_R,i}$. Thus $i=1$.

(2) In C, $s_1<s_2<\cdots<s_m$ implies that $b_{s_1,1}<b_{s_2,1}<\cdots<b_{s_m,1}$.

Furthermore, for all i in $[1,m]$, $\bar{\sigma}(\alpha(b_{s_i,l_{s_i}}))=b_{s_i,1}$ and $b_{s_i,1}$ is not the smallest half-edge of its face (see Lemma 3), so that $\alpha(b_{s_i,l_{s_i}})$ precedes $b_{s_i,1}$ in the ordered map C.

One then has in C, $\tilde{b}<\alpha(b_{s_1,l_{s_1}})<b_{s_1,1}<\alpha(b_{s_2,l_{s_2}})<b_{s_2,1}<\cdots<\alpha(b_{s_m,l_{s_m}})<b_{s_m,1}$.

Thus in C, the subpath from $\tilde{b}$ to $\alpha(b_{s_1,l_{s_1}})$ does not include any half-edge $\alpha(b_{s_i,l_{s_i}})$.

Once it has been proved that in $C'_{\mathscr{S},R}$, the subpath from $\tilde{b}$ to $\alpha(b_{s_1,l_{s_1}})$ does not include $\alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}})$, then it will follow from what precedes that in $C'_{\mathscr{S},R}$, the subpath from $\tilde{b}$ to $\alpha(b_{s_1,l_{s_1}})$ does not include any of the half-edges $\alpha(b_{s_i,l_{s_i}})$. Since

$$\overline{\sigma'}(a)=\begin{cases} b_{s_{i+1},1} & \text{if } a=\alpha(b_{s_i,l_{s_i}}) \quad \forall\, 1\leqslant i<m,\\ b_{\tilde{s}_R,1} & \text{if } a=\alpha(b_{s_m,l_{s_m}}),\\ b_{s_1,1} & \text{if } a=\alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}}),\\ \bar{\sigma}(a) & \text{if } a\in C, a\neq b_{s_i,l_{s_i}} \quad \forall\, 1\leqslant i\leqslant m,\\ \overline{\sigma_R}(a) & \text{if } a\in R, a\neq b_{\tilde{s}_R,l_{\tilde{s}_R}}, \end{cases}$$

it means that the subpath from $\tilde{b}$ to $\alpha(b_{s_1,l_{s_1}})$ in $C'_{\mathscr{S},R}$ is unchanged.

We now prove that the subpath of $C'_{\mathscr{S},R}$ from $\tilde{b}$ to $\alpha(b_{s_1,l_{s_1}})$ does not include the half-edge $\alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}})$.

Since $\overline{\sigma'}(\alpha(b_{s_m,l_{s_m}}))=b_{\tilde{s}_R,1}$ and $b_{\tilde{s}_R,1}$ is not the smallest half-edge of its face (see item 1 of this proof), $\alpha(b_{s_m,l_{s_m}})$ precedes $b_{\tilde{s}_R,1}$ in the path of $C'_{\mathscr{S},R}$.

Furthermore, from Lemma 5.1, $b_{\tilde{s}_R,1}<\alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}})$ since $\alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}})\in R$ and $b_{\tilde{s}_R,1}$ is the smallest half-edge of R in $C'_{\mathscr{S},R}$. Thus in $C'_{\mathscr{S},R}$, $\alpha(b_{s_1,l_{s_1}})<\alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}})$. □

The following technical lemma makes it possible to recover the vertices $s_1,\ldots,s_m,\tilde{s}_R$, which compose the vertex s, as will be shown in Lemma 10.

The notations of Definition 4 are used here.

Lemma 6. *In $C'_{\mathscr{S},R}$ the smallest half-edge among the half-edges of the vertex s is*

$$\sigma'(b_{s_1,l_{s_1}})=\begin{cases} b_{s_2,1} & \text{if } m>1,\\ b_{\tilde{s}_R,1} & \text{if } R\neq\{p\} \textit{ and } m=1,\\ b_{s_1,1} & \text{if } R=\{p\} \textit{ and } m=1. \end{cases}$$

Proof. (1) If $R=\{p\}$ and $m=1$ then $C=C'_{\mathscr{S},R}$, $s=s_1$ and thus, $\sigma'(b_{s_1,l_{s_1}})=b_{s_1,1}$ is the smallest half-edge among the half-edges of s.

(2) We assume that $R\neq\{p\}$ or $m\neq 1$. One has

$$\overline{\sigma'}(a)=\begin{cases} b_{s_{i+1},1} & \text{if } a=\alpha(b_{s_i,l_{s_i}}) \quad \forall\, i, 1\leqslant i<m,\\ b_{\tilde{s}_R,1} & \text{if } a=\alpha(b_{s_m,l_{s_m}}),\\ b_{s_1,1} & \text{if } a=\alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}}),\\ \bar{\sigma}(a) & \text{if } a\in C, a\neq\alpha(b_{s_i,l_{s_i}}) \quad \forall\, i, 1\leqslant i\leqslant m,\\ \overline{\sigma_R}(a) & \text{if } a\in R, a\neq\alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}}). \end{cases}$$

Let $\hat{b}$ be the smallest half-edge of face $\overline{\sigma}^*(b_{s_1,1})$ in C.

(a) In C, $b_{s_1,1}$ is the smallest half-edge of vertex s_1. From Lemma 3, since $s_1 \neq \tilde{s}_C$, $b_{s_1,1}$ is not the smallest half-edge of its face. It implies that there exists $j > 0$ such that $\bar{\sigma}^j(\hat{b}) = b_{s_1,1}$.
(b) Finally, we prove Lemma 6, that is: $\sigma'(b_{s_1,l_{s_1}})$ is the smallest half-edge of s in $C'_{\mathscr{S},R}$.

It follows from Lemma 5.2 that the subpath from $\tilde{b}$ to $\alpha(b_{s_1,l_{s_1}})$ in $C'_{\mathscr{S},R}$ is identical to the one in C. Thus $\alpha(b_{s_1,l_{s_1}}) = \bar{\sigma}^{j-1}(\hat{b}) = \overline{\sigma'}^{j-1}(\hat{b})$.

Furthermore, in C, the subpath from $\tilde{b}$ to $\alpha(b_{s_1,l_{s_1}})$ does not include s since $b_{s_1,1}$ is the smallest half-edge of the half-edges of s in C and $\alpha(b_{s_1,l_{s_1}})$ is smaller than $b_{s_1,1}$ in C (see the proof of Lemma 5.2). It is the same in $C'_{\mathscr{S},R}$.

Thus $\sigma'(b_{s_1,l_{s_1}}) = \overline{\sigma'}(\alpha(b_{s_1,l_{s_1}}))$ is the smallest half-edge of s in $C'_{\mathscr{S},R}$. $\square$

Now $b_{s_1,1}$ is the smallest half-edge of $\mathscr{S}$ in C. Its predecessor in the path of C is the half-edge $\alpha(b_{s_1,l_{s_1}})$ since $b_{s_1,1}$ is not the smallest half-edge of its face (see Lemma 3). In map $C'_{\mathscr{S},R}$, obtained from C and R by gluing together certain vertices of C and the root vertex of R in one vertex s, the successor of $\alpha(b_{s_1,l_{s_1}})$ becomes $b_{s_2,1}$, which is then the smallest half-edge of s in $C'_{\mathscr{S},R}$ reordered. If $b_{s_1,1}$ has been marked, one thus first retrieves the vertex s_1 which is detached from s and then recursively the vertices $s_2, \ldots, s_m$. Thus the pair of initial maps can be recovered from its derived map. A formal definition of this inverse operation, which will be called integration, is given in the next section.

3.2.2. Integration of a map

A topological operation of opening of a vertex into two vertices is introduced in order to define the integration of a map, which consists of the splitting of a vertex into several vertices. It will then be seen that to recover a pair of maps (C, R) and the subset of vertices of C if its derived map is known, one has to integrate this last map.

Definition 7 (*Topological operation of opening of a map with respect to a half-edge*). Let $C = (\sigma, \alpha, \tilde{b})$ be a map and b a half-edge of C. Let b_s be the smallest half-edge of the vertex $s = \sigma^*(b)$. The *opening* of C with respect to b consists of the splitting of the vertex s into two vertices s_1 and s_2 in the following way:

$$s = (b, \ldots, \sigma^{-1}(b_s), b_s, \ldots, \sigma^{-1}(b)) \to s_1 = (b, \ldots, \sigma^{-1}(b_s)) \\ \text{and } s_2 = (b_s, \ldots, \sigma^{-1}(b)).$$

It means that the following permutation $\widehat{\sigma_b}$ is applied to the half-edges of C: $\widehat{\sigma_b} = \tau\sigma$ with $\tau = (bb_s)$.

The result of the opening of C with respect to b is a map or a pair of maps:

(i) If $b_s \neq b$ and if the group generated by $(\widehat{\sigma_b}, \alpha, \tilde{b})$ acts transitively on the set of half-edges of C (i.e. $(\widehat{\sigma_b}, \alpha, \tilde{b})$ generates a map and not two disconnected maps), then a new map $\widehat{C_b} = (\widehat{\sigma_b}, \alpha, \tilde{b})$ is defined.

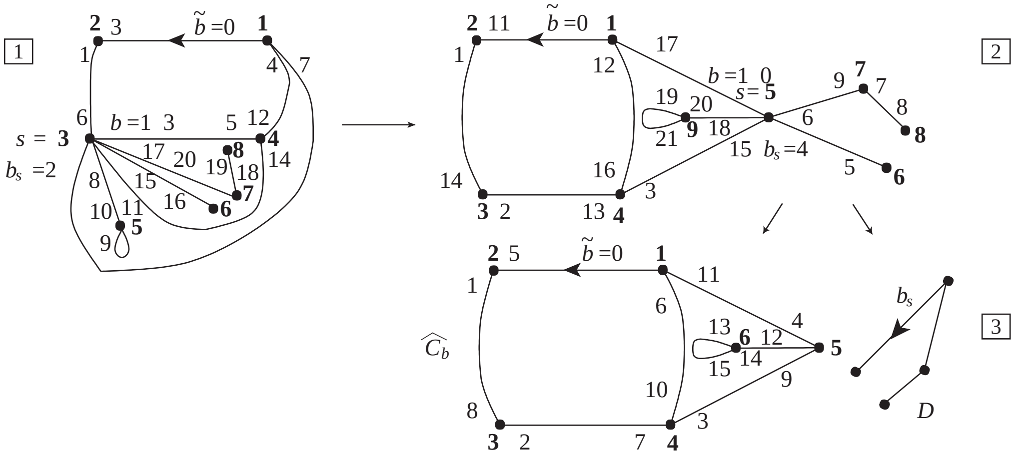

Fig. 2. Integration of map C with respect to the half-edge b: a pair of maps $(\widehat{C_b}, D)$ of $\mathscr{M}_2 \times \mathscr{M}$ is obtained.

(ii) Otherwise, a pair of maps $(\widehat{C_b}, D)$, $\widehat{C_b} = (\widehat{\sigma_b}, \alpha, \tilde{b})$, $D = (\widehat{\sigma_b}, \alpha, b_s)$, is obtained, D being the vertex-map $\{p\}$ if $b_s = b$.

Remark 8. If $s \neq \tilde{s}$, $\widehat{C_b} \in \mathscr{M}_2$.

The next definition explains that in order to integrate a map C with respect to a given half-edge b, one has to recursively apply this topological operation of opening of C until a pair of maps is obtained.

Definition 9 (*Integration of a map*). Let $C = (\sigma, \alpha, \tilde{b})$ be a map of $\mathscr{M}_2$, with root vertex $\tilde{s}$. Let $s \neq \tilde{s}$ be a vertex of C and $b \in s$. Let $\mathscr{S} = \emptyset$.

We say that a map C *is integrated* with respect to a half-edge b when the operation of the opening of C is recursively applied until case (ii) of Definition 7 is reached, that is:

- We denote by b_s the smallest half-edge of $\sigma^*(b)$; then C is opened with respect to b (see Definition 7).
- If this operation yields a map $\widehat{C_b}$ (see Fig. 2, drawing $\boxed{2}$), the vertex obtained after the opening, incident to b (the other obtained vertex is incident to b_s), is added to $\mathscr{S}$ and the opening operation starts again with $C \leftarrow \widehat{C_b}$ and $b \leftarrow b_s$.
- Otherwise, a pair of maps of $\mathscr{M}_2 \times \mathscr{M}$, $(\widehat{C_b}, D)$ is obtained (see Fig. 2, drawing $\boxed{3}$), and also a set of vertices of $\widehat{C_b}$, $\mathscr{S}$ with the added vertex of $\widehat{C_b}$ which was split from the root vertex of D (vertex of $\widehat{C_b}$ to which b belongs).

Lemma 10. *Let $C'_{\mathscr{S},R}$ be the derived map of a pair of maps (C, R) of $\mathscr{M}_2 \times \mathscr{M}$ with respect to a set of vertices $\mathscr{S}$ of C. We denote by b $(=b_{s_1,1}$ of Definition 4) the smallest half-edge of $\mathscr{S}$ in C. Integration of $C'_{\mathscr{S},R}$ with respect to b gives $(C, \mathscr{S}, R)$.*

Proof. With notations of Definitions 4 and 9, the map $C'_{\mathscr{S},R} = (\sigma', \alpha, \tilde{b})$ is integrated with respect to the half-edge $b_{s_1,1}$: $b{=}b_{s_1,1}$ and $b_s{=}b_{s_2,1}$ (from Lemma 6). The opening operation of the vertex s unglues the vertex s_1 from s and yields the map $\widehat{(C'_{\mathscr{S},R})}_b = (\widehat{\sigma'}_b, \alpha, \tilde{b})$:

$$s{=}(\underset{\substack{\uparrow\\ b}}{b_{s_1,1}}, \ldots, b_{s_1,l_{s_1}}, \underset{\substack{\uparrow\\ b_s.}}{b_{s_2,1}}, \ldots, b_{s_2,l_{s_2}}, \ldots, b_{s_m,1}, \ldots, b_{s_m,l_{s_m}}, b_{\tilde{s}_R,1}, \ldots, b_{\tilde{s}_R,l_{\tilde{s}_R}}).$$

Two vertices are obtained: a vertex $s_1 = (b_{s_1,1}, \ldots, b_{s_1,l_{s_1}})$ and a vertex $s = (b_{s_2,1}, \ldots, b_{s_2,l_{s_2}}, \ldots, b_{s_m,1}, \ldots, b_{s_m,l_{s_m}}, b_{\tilde{s}_R,1}, \ldots, b_{\tilde{s}_R,l_{\tilde{s}_R}})$. One has: $\widehat{\sigma'}_b = \tau_{12}\sigma'$.

Thus, $(\sigma_2 = \widehat{\sigma'}_b, \alpha, \tilde{b}) = C'_{\{s_2,\ldots,s_m\},R}$ and $\mathscr{S} = \{s_1\}$. One successively obtains maps $C'_{\{s_i,\ldots,s_m\},R} = (\sigma_i = \tau_{i-1i}\sigma_{i-1}, \alpha, \tilde{b})$ for $\tau_{i-1i} = (b_{s_{i-1},1}b_{s_i,1})$, and $\mathscr{S} = \{s_1, \ldots, s_{i-1}\}$, with $3 \leqslant i \leqslant m$. Applying for the last time the topological operation of opening of $s = (b_{s_m,1}, \ldots, b_{s_m,l_{s_m}}, b_{\tilde{s}_R,1}, \ldots, b_{\tilde{s}_R,l_{\tilde{s}_R}})$ to $C'_{\{s_m\},R}$, two disconnected maps, $C = (\sigma, \alpha, \tilde{b})$ and $R{=}(\sigma, \alpha, b_{\tilde{s}_R,1})$, are recovered and also $\mathscr{S}{=}\{s_1, \ldots, s_m\}$. One has: $\sigma{=}\tau_{Rm}\tau_{mm-1}\ldots\tau_{12}\sigma'{=}\delta\sigma'$ with $\delta = \gamma^{-1}$ (see Definition 4). $\square$

4. Generalized Dyck equation on maps of indeterminate genus

The well-known Dyck equation on trees is based on a one-to-one correspondence between the set of rooted planar trees $\mathscr{A}$ minus the vertex-tree, and $\mathscr{A}^2$. In Section 4.1, an equation generalizing the Dyck equation to rooted maps counted independent of genus is given. This equation is equivalent to an equation on sets which is derived in Section 4.1. A proof of the equation on sets is given in Section 4.2. The topological operations introduced in Section 3.2 will be used for this demonstration.

4.1. Generalized Dyck equations

The equation on sets is given as a bijection between the set of rooted maps of indeterminate genus, $\mathscr{M}$, and the set of pairs of maps of $\mathscr{M}$, where in one of these maps a subset (possibly empty) of its vertices is selected. Eq. (2) is then an expression of this bijection in terms of generating functions.

For any map of $\mathscr{M}$, we denote by $\mathscr{V}_M$ the set of vertices of M and $\mathscr{P}(\mathscr{V}_M)$ the set of all subsets of $\mathscr{V}_M$.

Theorem 11 (*Equation on sets*).

$$\mathscr{M} \leftrightarrow \{p\} \cup \left[\bigcup_{M\in\mathscr{M}} M \times \mathscr{P}(\mathscr{V}_M)\right] \times \mathscr{M}. \tag{1}$$

The proof of this theorem is given in Section 4.2.

The expression of this bijection in terms of generating functions yields a generalized Dyck equation generalizing the Dyck equation on trees.

We denote by y the variable whose exponent enumerates the vertices of a map of $\mathscr{M}$ and by z the variable whose exponent enumerates the edges of a map of $\mathscr{M}$ and $M(y, z)$ the generating function of rooted maps of indeterminate genus.

We obtain the following corollary:

Corollary 12 (*Generalized Dyck equation*).

$$M(y, z) = y + zM(y, z)M(y + 1, z). \tag{2}$$

4.2. Proof of Theorem 11

A bijection between maps of $\mathscr{M}$, different from the vertex-map and $(\bigcup_{M\in\mathscr{M}} M \times \mathscr{P}(\mathscr{V}_M)) \times \mathscr{M}$ is described, which means between maps of $\mathscr{M}$ and maps of $\mathscr{I}$ in which for each map I of $\mathscr{I}$, a set $\mathscr{S}$ of vertices of the submap containing the terminal vertex of the root half-edge, $\mathrm{Left}(I)$, has been selected. In fact, $\mathscr{I}$ is in one-to-one correspondence with $\mathscr{M}^2$, since to each map I of $\mathscr{I}$, a pair of maps of $\mathscr{M}^2$, $(\mathrm{Left}(I), \mathrm{Right}(I))$, can be associated. Furthermore, the set of pairs $(\mathrm{Left}(I), \mathscr{S})$ is the set $\bigcup_{M\in\mathscr{M}} M \times \mathscr{P}(\mathscr{V}_M)$.

Lemma 13 (*Bijection of Theorem 11*). *There is a one-to-one correspondence between $\mathscr{M}$ and the set of pairs $(I, \mathscr{S})$, in which I is a map of $\mathscr{I}$ and $\mathscr{S}$ is a set of vertices of* $\mathrm{Left}(I)$, *possibly empty*.

Proof. Integration of a map with respect to a half-edge b makes it possible to recover a pair of maps as well as a set of vertices of one of the maps thus obtained. Thus, when a derived map I' is obtained, to retrieve the original map one has to store the half-edge b. To do this, if the root vertex of I' is incident only to the root half-edge, then it is sufficient to glue the root half-edge just before b in order to obtain a map M of $\mathscr{M}$.

Starting with a map I of $\mathscr{I}$ in which a set $\mathscr{S}$ of vertices of $\mathrm{Left}(I)$ has been selected, we will first show how to obtain a map M of $\mathscr{M}$, and then how to recover map I and its set of vertices $\mathscr{S}$ from M.

Let $I = (\sigma, \alpha, \tilde{b})$ be a map of $\mathscr{I}$ with root vertex $\tilde{s}_I$ (see Fig. 3). We denote by I_L the map I minus $\mathrm{Right}(I)$, with the same root half-edge as I, and $\tilde{s}_R = \tilde{s}_I \backslash \{\tilde{b}\}$ the root vertex of $\mathrm{Right}(I)$.

Stage 1: Derivation of $(I_L, \mathscr{S}, \mathrm{Right}(I))$: Let $\mathscr{S}$ be a subset of vertices of $\mathrm{Left}(I_L) = \mathrm{Left}(I)$.

If $\mathscr{S}$ is not empty, let $\{s_1, \dots, s_m\}$ be m distinct vertices of $\mathscr{S}$ such that $s_1 < \cdots < s_m$. For all i in $[1, m]$, let $(b_{s_i,1}, \dots, b_{s_i,l_{s_i}}) = \sigma^*(b_{s_i,1})$ be the half-edges of the initial vertex s_i, in which $b_{s_i,1}$ is the smallest half-edge of s_i. Let $(b_{\tilde{s}_R,1}, \dots, b_{\tilde{s}_R,l_{\tilde{s}_R}}) = \tilde{s}_R$, with $b_{\tilde{s}_R,1} = \sigma(\tilde{b})$. Let $I' = (I_L)'_{\mathscr{S},\mathrm{Right}(I)} = (\sigma', \alpha, \tilde{b})$ be the derived map of $(I_L, \mathrm{Right}(I))$ with respect to $\mathscr{S}$. We recall that the vertices of $\mathscr{S} \cup \{\tilde{s}_R\}$ are joined into one vertex $s_d$in the following

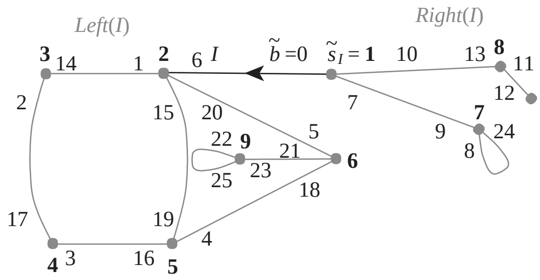

Fig. 3. Map of $\mathscr{I}$.

way (see Fig. 4):

$$s_d = \left(\underbrace{b_{s_1,1}, \ldots, b_{s_1,l_{s_1}}}_{s_1}, \underbrace{b_{s_2,1}, \ldots, b_{s_2,l_{s_2}}}_{s_2}, \ldots, \underbrace{b_{s_m,1}, \ldots, b_{s_m,l_{s_m}}}_{s_m}, \underbrace{b_{\tilde{s}_R,1}, \ldots, b_{\tilde{s}_R,l_{\tilde{s}_R}}}_{\tilde{s}_R} \right)$$
$$= \sigma'^*(b_{s_1,1}).$$

If $\mathscr{S}$ is empty, i.e. $m = 0$, then $I' = I$.

Stage 2: Labelling $b_{s_1,1}$ and obtaining a map of $\mathscr{M}$.

- If $\mathscr{S} = \emptyset$, $I' = I$ ((I_L, Right(I)) has been derived with respect to no vertex) and $M = I' = I$.
- Otherwise a map $M = (\sigma_M, \alpha, \tilde{b})$ is built (see Fig. 4), gluing the root vertex of $\tilde{b}$ to the vertex s_d in the following way:

$$\left(\underbrace{b_{s_1,1}, \ldots, b_{s_1,l_{s_1}}}_{s_1}, \underbrace{b_{s_2,1}, \ldots, b_{s_2,l_{s_2}}}_{s_2}, \ldots, \underbrace{b_{s_m,1}, \ldots, b_{s_m,l_{s_m}}}_{s_m}, \underbrace{b_{\tilde{s}_R,1}, \ldots, b_{\tilde{s}_R,l_{\tilde{s}_R}}}_{\tilde{s}_R}, \tilde{\mathbf{b}} \right)$$
$$= \sigma_M^*(\tilde{b}).$$

The following permutation is then applied to the half-edges of I': $\sigma_M = (\tilde{b} b_{s_1,1})\sigma'$.

Remark 14. If $\mathscr{S} = \emptyset$, $M = I$ and if I is a tree, then M is a tree.

Recovering $(I, \mathscr{S})$ from M. If the map M obtained belongs to $\mathscr{I}$, then $\mathscr{S}$ was empty and now $M = I$. Thus to retrieve I from M, nothing has to be done. We remark that, thanks to Remark 14, when we restrict ourselves to the case of trees, one recovers the decomposition induced by the Dyck equation on trees.

We assume that M does not belong to $\mathscr{I}$. Then $\sigma_M(\tilde{b}) = b_{s_1,1}$. In the map I, $b_{s_1,1}$ is the smallest half-edge among the half-edges incident to the vertices of $\mathscr{S}$. The conditions of Lemma 10 are satisfied, and one can apply this lemma. Thus, to recover $(I, \mathscr{S})$ from M,

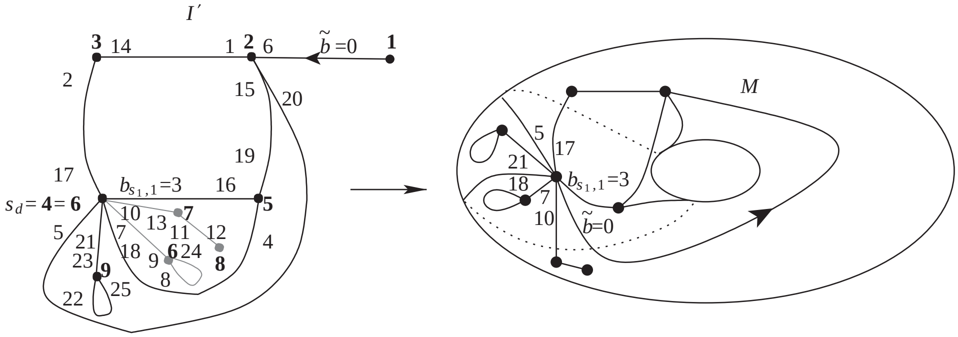

Fig. 4. Map $I' = (I_L)'_{\{\mathbf{4},\mathbf{6}\},\mathrm{Right}(I)}$ and the map M of $\mathscr{M}$ obtained from the map I' and its half-edge $b_{s_1,1}$ (I' and M have not been reordered).

one has to:

- unglue $\tilde{b}$ from the root vertex,
- integrate this new map M_1 with respect to $b_{s_1,1} = \sigma_M(\tilde{b})$. From Lemma 10, one retrieves I_L and $\mathrm{Right}(I)$, respectively, rooted in $\tilde{b}$ and $b_{\tilde{s}_R,1}$, and $\mathscr{S}$.
- Then $\mathrm{Right}(I)$ is glued to the root vertex of I_L, such that $\sigma(\tilde{b}) = b_{\tilde{s}_R,1}$. □

5. Bijection between maps of indeterminate genus and multilabel trees

The operation that makes it possible to prove Theorem 11 transforms a map of $\mathscr{M}$ into a map with a bridge root edge in which a subset of its vertices has been selected. If this operation is iterated on the successive submaps incident to the two vertices incident to the bridge half-edge, and if the subset of vertices associated with each map is labelled (one distinct label for each subset), the initial map is transformed into a tree whose vertices can be labelled with several labels, following repartition rules to be defined later. One then obtains what we will call a *multilabel tree*.

In Section 5.1, we give the definition of a multilabel tree and in Section 5.2, we prove the one-to-one correspondence between maps of $\mathscr{M}$ and multilabel trees. This bijection leads to a coding of maps by words of a language, as shown in Section 5.3.

5.1. Multilabel trees

We give the definition of a multilabel tree. We then define a one-to-one correspondence in Section 5.2 between multilabel trees and maps of indeterminate genus. These multilabel trees are trees whose vertices can be labelled with several labels, following repartition rules that will be defined. The order relations given in Section 3.1 are applied to multilabel trees. An order on half-edges and vertices is thus established in a classical depth-first descent of the tree. We note that the smallest half-edge of a vertex is also its left son in the tree structure, since a tree has only one face.

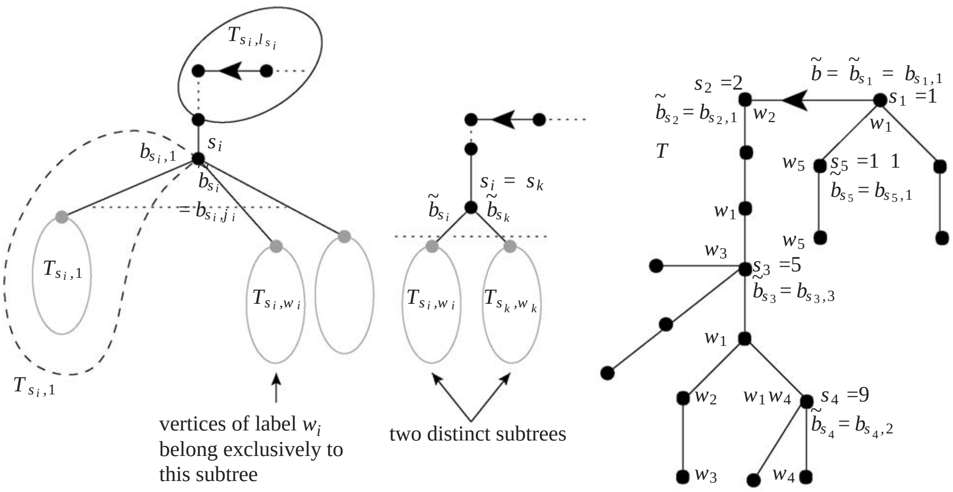

Illustration of rule 2a   Illustration of rule 2b   Illustration of rule 3 for i=1 and j=4

Fig. 5. Illustration of Definition 15.

Definition 15 (*Multilabel tree*). Let $T = (\sigma, \alpha, \tilde{b})$ be a rooted tree. Let $\mathscr{W} = \{w_1, \ldots, w_n\}$ be a set of n distinct labels, possibly empty ($n \geqslant 0$). Each vertex of T can be labelled by 0 to n labels. For all i in $[1, n]$, we denote the smallest vertex of T of label w_i by s_i.

T is a multilabel tree (see Fig. 5) if T complies with the following rules:

(1) each label of W is assigned to at least two distinct vertices of T;
(2) let $(b_{s_i,1}, \ldots, b_{s_i,l_{s_i}}) = \sigma^*(b_{s_i,1})$ be the half-edges of the initial vertex s_i, where $b_{s_i,1}$ is the smallest half-edge of $\sigma^*(b_{s_i,1})$, the leftmost half-edge of s_i. The half-edges $b_{s_i,j}$, $1 \leqslant j < l_{s_i}$, are the half-edges whose terminal vertices are sons of s_i, and $b_{s_i,l_{s_i}}$ is the half-edge which goes up towards the father of s_i. Let $T_{s_i,j}$ be the subtree of T containing the terminal vertex of $b_{s_i,j}$, rooted in $\bar{\sigma}(b_{s_i,j})$ and let $\overline{T}_{s_i,j}$ be the tree composed of $T_{s_i,j}$ and of the half-edge $b_{s_i,j}$ which is its root half-edge. Then:
   (a) there is a single j_i such that in T, w_i labels s_i and exclusively vertices of T_{s_i,j_i}. We denote this subtree by $T_{s_i,w_i} = T_{s_i,j_i}$, $\tilde{b}_{s_i} = b_{s_i,j_i}$ its root half-edge and $\overline{T}_{s_i,w_i} = \overline{T}_{s_i,j_i}$;
   (b) for all k in $[1, n]$, $k \neq i$, if $s_i = s_k$ then $\overline{T}_{s_i,w_i} \cap \overline{T}_{s_k,w_k} = \emptyset$.
(3) For all distinct labels w_i and w_j, if there is a vertex s of labels w_i and w_j where s_i is smaller than s_j, then $s = s_j$ and s is the only vertex of label w_j which is also of label w_i.

We will say that two multilabel trees are *isomorphic* if one can be obtained from the other by a permutation on its labels. An isomorphism class of multilabel trees will simply be called *multilabel tree*.

Let $\mathscr{T}$ be the set of multilabel trees.

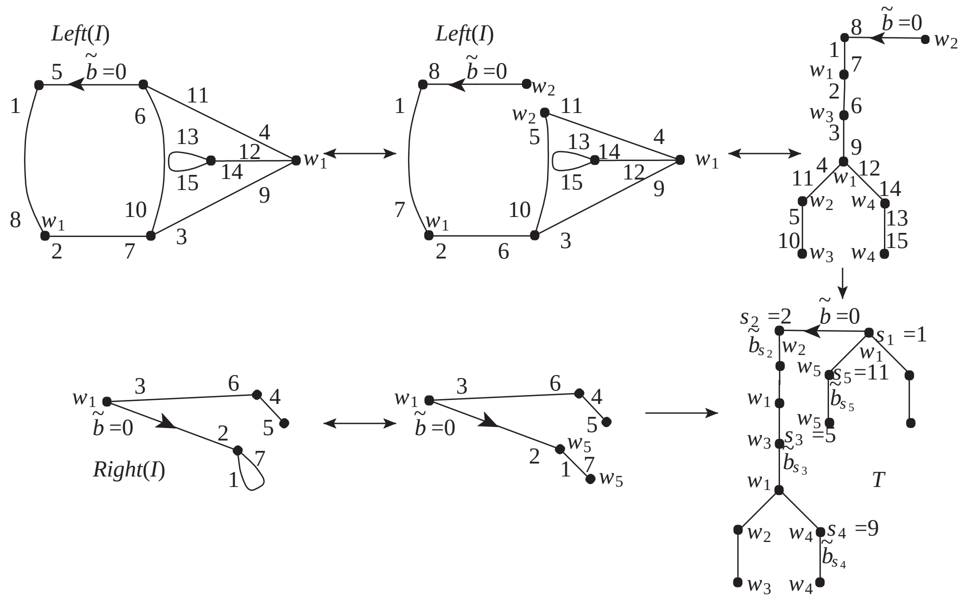

Fig. 6. The multilabel tree associated with the map M of Fig. 4.

Remark 16. If T is a multilabel tree with m vertices and n distinct labels, then $n < m$.

5.2. Bijection between $\mathscr{M}$ and $\mathscr{T}$

Theorem 17. *The set of rooted maps of indeterminate genus with n edges is in bijection with the family of multilabel trees with n edges.*

Proof. In order to simplify our notations, a map whose vertices can be labelled (by several labels) will also be called a map.

Let M be a map of $\mathscr{M}$ not reduced to the vertex-map and let w_1 be a label. Let $w = w_1$. A map T of $\mathscr{T}$ is obtained from map M (see Fig. 6) in the following way:

(1) (a) If M deprived of its labels does not belong to $\mathscr{I}$, one applies to M the decomposition induced by Lemma 13, which transforms bijectively a map of $\mathscr{M}$ into a pair $(I, \mathscr{S})$, where $I \in \mathscr{I}$ and $\mathscr{S}$ is a set of vertices of $\mathrm{Left}(I)$. Then one assigns label w to the vertices resulting from the partition of the root vertex, $\tilde{s}_M$, of M, i.e. to the vertices of $\mathscr{S} \cup \{\tilde{s}_I\}$, where $\tilde{s}_I$ is the root vertex of the map I thus obtained. If labels are not taken into account, then $I \in \mathscr{I}$. Let $\mathscr{W}_{\tilde{s}_M}$ be the set of labels which label the root vertex $\tilde{s}_M$ of M. Then in I, the set $\mathscr{W}_{\tilde{s}_I}$ of labels assigned to $\tilde{s}_I$ is equal to $\mathscr{W}_{\tilde{s}_M} \cup \{w\}$. Labels of $\mathscr{W}_{\tilde{s}_M}$ are not transferred to the other vertices resulting from $\tilde{s}_M$.

(b) Otherwise, M is renamed I.

(2) If $I \notin \mathscr{T}$, let w_{left} and w_{right} be two distinct labels, also distinct from all the labels already labelling I. One begins again at stage 1a with $M = \text{Left}(I)$, $w = w_{\text{left}}$ and $M = \text{Right}(I)$, $w = w_{\text{right}}$.

From Lemma 13, it follows that each stage of the transformation of a map of $\mathscr{M}$ into a multilabel tree is bijective.

By construction, T follows all the rules of Definition 15, and $T \in \mathscr{T}$, since:

- 1 is satisfied since a label w is assigned to the vertices resulting from the same vertex s.
- 2a is satisfied since if M deprived of its labels does not belong to $\mathscr{I}$, the root half-edge of M is unglued from the root vertex and a map M_1 is obtained. Thus all the vertices to be labelled belong to $\text{Left}(M_1)$, the root vertex excluded.
- 2b is satisfied since after application of the transformation induced by Lemma 13 to a map of $\mathscr{M}$, its root half-edge becomes a bridge.
- 3 is satisfied according to item 1a above. □

5.3. Application: a language coding maps of indeterminate genus

In this section we present a language coding rooted maps. The equation defining this language is a generalization of the well-known equation on Dyck words. In fact, this language codes multilabel trees and thus, by bijection, rooted maps. The proof is easily obtained by induction and is not given here.

In order to clarify the significance of each letter of the alphabet of the language that we present, we need to give a definition.

Definition 18 (*Twin labels*). Two labels w and w' of a tree of $\mathscr{T}$ are *twin* if there is a vertex of T labelled by these two labels or if there is a subsequence of labels of T, $w_1 = w, w_2, \ldots, w_n = w'$ such that for all i in $[1, n]$, w_i and w_{i+1} label the same vertex. One thus defines equivalence classes of labels, where two labels are in the same class if they are twin.

We denote by c the variable coding a half-edge whose opposite half-edge is not coded, $\bar{c}$ the variable coding an half-edge whose opposite half-edge is coded, y the variable coding a vertex in case of a map or, in case of a multilabel tree, a vertex not labelled or the smallest vertex among the vertices having the same or a twin label, and y_i, $i \geqslant 1$, the variable coding a vertex of label w_i (with $w_i \neq w_j$ if $i \neq j$) of a multilabel tree. In a rooted map, y_i codes the half-edges belonging to a subset of the set of half-edges of initial vertex s_i, for a given vertex s_i of arity strictly superior to 1 (s_i can be equal to s_j if $i \neq j$).

Theorem 19. *The set of rooted maps is coded by the language $L_\infty = \lim_{n\to\infty} L_n$, where L_n represents the language coding rooted maps with at most n edges and is defined in* *the following way*:

$$L_0(y, c, \bar{c}) = y, \tag{3}$$

$$\begin{aligned} &L_n(y, y_1, \ldots, y_n, c, \bar{c}) \\ &\quad = y + \sum_{\substack{m_1 \in L_{n-1}(y+y_n, y_1, \ldots, y_{n-1}, c, \bar{c}) \\ m_2 \in L_{n-1}(y, y_1, \ldots, y_{n-1}, c, \bar{c}),}} c m_1 \bar{c} m_2 (1 - \varepsilon_{m_1,n} + y_n \varepsilon_{m_1,n}) \delta_{m_1,m_2,n}, \end{aligned} \tag{4}$$

where:

- $\varepsilon_{m_1,n} = \begin{cases} 1 & \textit{if } y_n \in m_1, \\ 0 & \textit{otherwise}. \end{cases}$
- $\delta_{m_1,m_2,n} = \begin{cases} 1 & \textit{if the number of occurrences of } c \textit{ in } c\ m_1\ \bar{c}\ m_2 \leqslant n, \\ & \textit{and } \nexists 1 \leqslant k \leqslant n / y_k \in m_1 \textit{ and } y_k \in m_2, \\ 0 & \textit{otherwise}. \end{cases}$

6. Colored maps

We recall the definition of an n-colored map and give the generalized Dyck equations for n-colored rooted maps of indeterminate genus in Section 6.1 and the bijection between these maps and n-colored rooted labelled trees in Section 6.2. This one-to-one correspondence leads to a language coding n-colored rooted maps, described in Section 6.3. The proofs follow the same sketches as those for rooted maps and are not given here.

Definition 20 (n-*colored map*). An orientable rooted n-*colored map* ($n > 1$) is a rooted map, where a maximum of n colors are used to color the vertices and such that each edge is incident to two vertices of different colors.

The property "n-colored" is compatible with the equivalence relation whose classes are the rooted maps.

6.1. Generalized Dyck equations

The equation on sets is given as a bijection between the set of n-colored rooted maps of indeterminate genus with a root vertex of color i, $\mathscr{M}_{n,i}$, and the set of pairs of maps of $\bigcup_{j=1, j\neq i}^{n} \mathscr{M}_{n,j} \times \mathscr{M}_{n,i}$, where in one of these maps a subset (possibly empty) of its vertices of color i is selected (see Fig. 7). Eq. (5) is then an expression of this bijection in terms of generating functions.

For any map M of $\mathscr{M}_{n,i}$, we denote by $\mathscr{V}_{i,M}$ the set of vertices of color i of M and $\mathscr{P}(\mathscr{V}_{i,M})$ the set of all subsets of $\mathscr{V}_{i,M}$.

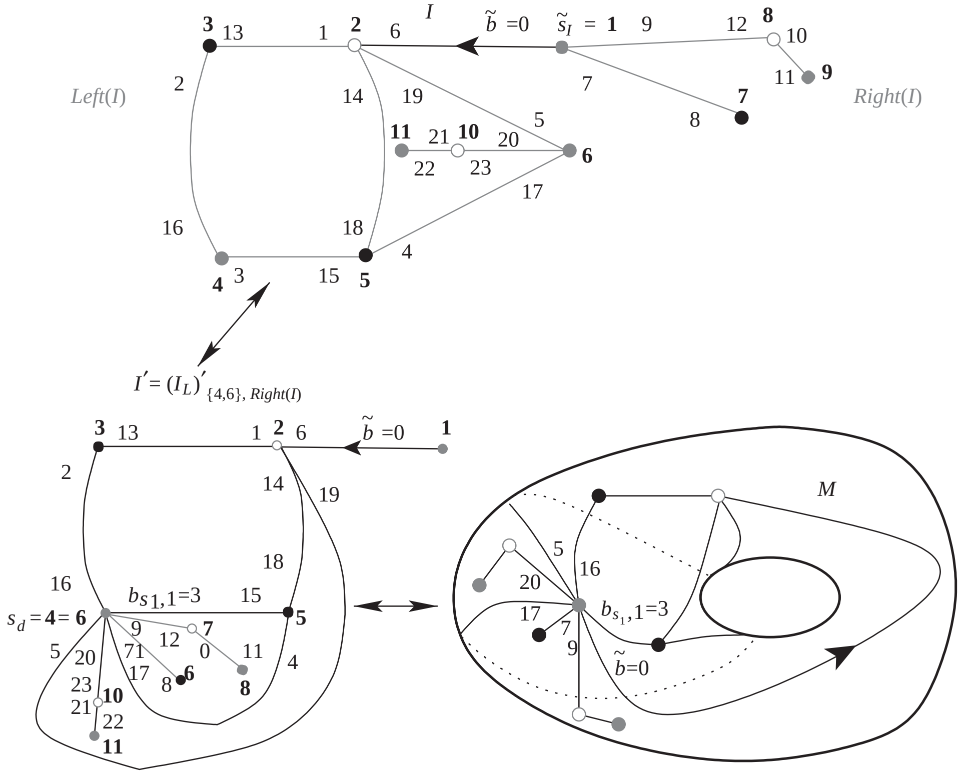

Fig. 7. Bijective transformation of a map I of $\mathscr{I}_{n,i}$ into a map M of $\mathscr{M}_{n,i}$ (I' and M have not been reordered).

Theorem 21.

$$\mathscr{M}_{n,i} \leftrightarrow \{p_i\} \bigcup_{j=1,\, j\neq i}^{n} \left[\bigcup_{M\in\mathscr{M}_{n,j}} M \times \mathscr{P}(\mathscr{V}_{i,M}) \right] \times \mathscr{M}_{n,i}.$$

Let $\mathscr{I}_n$ be the set $\{1,\ldots,n\}$. Let $M_{n,i}$ (resp. M_i), $i \in \mathscr{I}_n$, be the generating function of maps of $\mathscr{M}_{n,i}$ enumerated by vertices (resp. vertices of color $j \in \mathscr{I}_n$) and half-edges whose initial vertex is of color $j \in \mathscr{I}_n$. Let c_i, $i \in \mathscr{I}_n$, be the variable whose exponent represents the number of half-edges with an initial vertex of color i. Let y be the variable whose exponent represents the number of vertices of the map. Henceforth, we will write $M_{n,i}$ for $M_{n,i}(y, c_1, \ldots, c_n)$ and $M_i(u)$ for $M_i(u; c_1, \ldots, c_n)$ with $u = (u_j)_{1\leqslant j\leqslant n}$.

Corollary 22 (*Generalized Dyck equation*).

$$M_{n,i} = y + c_i M_{n,i} \sum_{j=1,\, j\neq i}^{n} c_j M_j(v) \tag{5}$$

with $v = (v_j)_{1\leqslant j\leqslant n} = (y + \delta_{ij})_{1\leqslant j\leqslant n}$ *and* δ_{ij} *is the Kronecker symbol.*

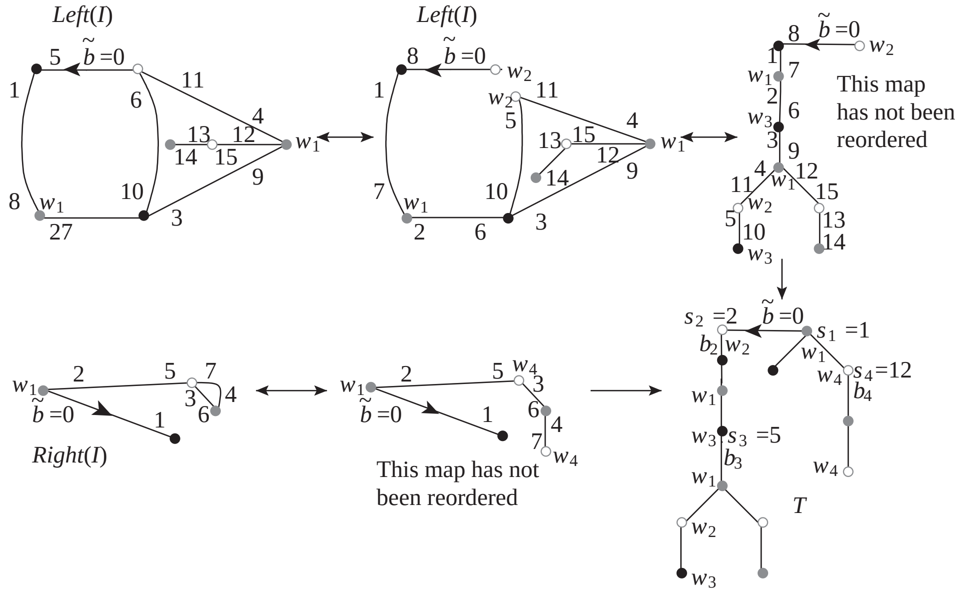

Fig. 8. The multilabel n-colored tree associated with the map M of Fig. 7.

6.2. Bijection between n-colored maps and multilabel n-colored trees

Definition 23 (*Multilabel n-colored tree*). A *multilabel n-colored tree* is an n-colored tree following the rules of Definition 15 such that the vertices with the same label are of the same color.

Let $\mathscr{T}_{n,i}$ be the set of multilabel n-colored trees, with a root vertex of label i.

Theorem 24. *The set of rooted n-colored maps with a root vertex of color i and p edges is in bijection with the family of multilabel n-colored trees with a root vertex of color i and p edges* (*see Fig*. 8).

6.3. Application: a language coding n-colored maps of indeterminate genus

In this section, we present a language coding rooted n-colored maps. In fact, this language codes multilabel n-colored trees and thus, by bijection, rooted n-colored maps.

We denote by e (resp. $\bar{e}$) the variable coding a half-edge whose opposite half-edge is not coded (resp. is coded), v_i the variable coding a vertex of color i in case of maps, and in case of multilabel n-colored trees a vertex of color i without any label or the smallest vertex of color i of a same or twin label. Let y_j, $j \geqslant 1$, be the variable coding a vertex of label w_j (with $w_j \neq w_k$ if $j \neq k$) of a multilabel n-colored tree. In a rooted n-colored map, y_j

codes the half-edges belonging to a subset of the set of half-edges of the initial vertex s_j, for a given vertex s_j of arity strictly superior to 1 (s_j can be equal to s_k if $j \neq k$).

We denote $(v_j)_{1 \leqslant j \leqslant n}$ by $\vec{v}$ and $(v_j + y_p\delta_{jq})_{1 \leqslant j \leqslant n}$ by $\vec{v}_{q,p}$.

Theorem 25. *The set of rooted n-colored maps with a root vertex of color i is coded by the language $L_{\infty,i} = \lim_{p\to\infty} L_{p,i}$, where $L_{p,i}$ represents the language coding maps of $\mathscr{M}_{n,i}$ with at most p edges and is defined in the following way*:

$$\begin{aligned} &L_{p,i}(\vec{v}, y_1, \ldots, y_p, e, \bar{e}) \\ &= v_i + e \sum_{j=1, j\neq i}^{n} \sum_{\substack{m_1 \in L_{p-1,j}(\vec{v}_{i,p}, y_1, \ldots, y_{p-1}, e, \bar{e}) \\ m_2 \in L_{p-1,i}(\vec{v}, y_1, \ldots, y_{p-1}, e, \bar{e}),}} m_1 \bar{e} m_2 (1 - \varepsilon_{m_1,p} + y_p \varepsilon_{m_1,p}) \delta_{m_1,m_2,p}, \end{aligned} \tag{6}$$

$$L_0(\vec{v}, e, \bar{e}) = v_i, \tag{7}$$

where:

$$\varepsilon_{m_1,p} = \begin{cases} 1 & \textit{if } y_p \in m_1, \\ 0 & \textit{otherwise}. \end{cases}$$

$$\delta_{m_1,m_2,p} = \begin{cases} 1 & \textit{if (the number of occurrences of } e \textit{ in } e m_1 \bar{e} m_2) \leqslant p \\ & \textit{and } \nexists 1 \leqslant k \leqslant p / y_k \in m_1 \textit{ and } y_k \in m_2, \\ 0 & \textit{otherwise}. \end{cases}$$

7. Conclusion

The one-to-one correspondences between maps and multilabel trees established here raise many questions. Can new equations on generating functions of families of maps be determined? Could they lead to new enumeration formulas of these families? These bijections can be specialized to planar maps [15] and it would be interesting to see whether it can also be done to maps of genus g, $g > 0$. It would also be interesting to see what kind of information we can obtain from the coding of maps presented here. It is straightforward to deduce the number of vertices and edges of a map from its associated word but we have not yet searched for other information.

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